

# OSCILLATION AND THE MEAN ERGODIC THEOREM

JEREMY AVIGAD AND JASON RUTE

**ABSTRACT.** Let  $\mathcal{B}$  be a uniformly convex Banach space, let  $T$  be a nonexpansive linear operator, and let  $A_n x$  denote the ergodic average  $\frac{1}{n} \sum_{i=0}^{n-1} T^i x$ . A generalization of the mean ergodic theorem due to Garrett Birkhoff asserts that the sequence  $(A_n x)$  converges, which is equivalent to saying that for every  $\varepsilon > 0$ , the sequence has only finitely many fluctuations greater than  $\varepsilon$ . Drawing on calculations by Kohlenbach and Leuştean [18], we provide a uniform bound on the number of fluctuations that depends only on  $\rho := \|x\|/\varepsilon$  and a modulus  $\eta$  of uniform convexity for  $\mathcal{B}$ . Specifically, we show that the sequence of averages  $(A_n x)$  has  $O(\rho^2 \log \rho \cdot \eta(1/(8\rho))^{-1})$ -many  $\varepsilon$ -fluctuations. The proof is fully explicit, providing a uniform, quantitative, and constructive formulation of the mean ergodic theorem for uniformly convex spaces.

## 1. INTRODUCTION

A Banach space  $\mathcal{B}$  is said to be *uniformly convex* if for every  $\varepsilon \in (0, 2]$  there exists a  $\delta \in (0, 1]$  such that for all  $x, y \in \mathcal{B}$ , if  $\|x\| \leq 1$ ,  $\|y\| \leq 1$ , and  $\|x - y\| \geq \varepsilon$ , then  $\|(x + y)/2\| \leq 1 - \delta$ . Any function  $\eta(\varepsilon)$  returning such a  $\delta$  for each  $\varepsilon$  is called a *modulus of uniform convexity*.

Let  $\mathcal{B}$  be a uniformly convex Banach space, and let  $T$  be a nonexpansive linear operator, that is, a linear map satisfying  $\|Tx\| \leq \|x\|$  for every  $x \in \mathcal{B}$ . For each  $n \geq 1$ , let  $A_n x$  denote the ergodic average  $\frac{1}{n} \sum_{i=0}^{n-1} T^i x$ . A generalization of the mean ergodic theorem due to Garrett Birkhoff [3] implies that the sequence  $(A_n x)$  converges. This is equivalent to saying that for every  $\varepsilon > 0$ , the sequence has only finitely many fluctuations greater than  $\varepsilon$ . Here we show that there is an explicit bound on the number of fluctuations that depends only on  $\|x\|$ ,  $\varepsilon$ , and a modulus of uniform convexity for  $\mathcal{B}$ . In particular, the bound is entirely independent of  $\mathcal{B}$ ,  $T$ , and  $x$ .

Let us be more precise. If  $(a_n)$  is any finite or infinite sequence of elements of  $\mathcal{B}$  and  $\varepsilon > 0$ , we will say that  $(a_n)$  *admits  $k$   $\varepsilon$ -fluctuations* if there are  $i_1 \leq j_1 \leq \dots \leq i_k \leq j_k$  such that, for each  $u = 1, \dots, k$ ,  $\|a_{j_u} - a_{i_u}\| \geq \varepsilon$ . (This terminology is used in [13]. Some authors use the term “jumps” instead of “fluctuations”; see, for example, [11].)

**Theorem 1.1.** *Let  $\mathcal{B}$  be any uniformly convex Banach space with modulus of uniform convexity  $\eta$ , and let  $T$  be any nonexpansive linear operator. For any  $x$  in  $\mathcal{B}$  and  $\varepsilon > 0$ , let  $\rho = \|x\|/\varepsilon$ . Then the sequence of ergodic averages  $(A_n x)$  admits at most  $O(\rho^2 \log \rho \cdot \eta(1/(8\rho))^{-1})$ -many  $\varepsilon$ -fluctuations. In the case where  $\eta(\varepsilon)$  can be written in the form  $\varepsilon \cdot \tilde{\eta}(\varepsilon)$  where  $\tilde{\eta}$  is nondecreasing, the conclusion holds with  $\eta$  replaced by  $\tilde{\eta}$ .*

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In the special case where  $\mathcal{B}$  is a Hilbert space, Jones, Ostrovskii, and Rosenblatt [10] provide a stronger result, with the following “square function” inequality:

**Theorem 1.2.** *Let  $\mathcal{H}$  be a Hilbert space and let  $T$  be any nonexpansive linear operator. Then for any  $x$  in  $\mathcal{H}$  and any sequence  $n_1 \leq n_2 \leq n_3 \leq \dots$ ,*

$$\left( \sum_{k=1}^{\infty} |A_{n_{k+1}}x - A_{n_k}x|^2 \right)^{1/2} \leq 25\|x\|.$$

Since any  $\varepsilon$ -fluctuation contributes  $\varepsilon^2$  to the sum above, Theorem 1.2 implies that the sequence of ergodic averages admits at most  $625\rho^2$ -many  $\varepsilon$ -fluctuations, whereas Theorem 1.1 yields the weaker bound  $O(\rho^3 \log \rho)$ . The proof of Theorem 1.2 in [10] relies on the spectral theorem, and does not seem to carry over to uniformly convex Banach spaces.

We will see below that Theorem 1.1 follows fairly straightforwardly from the subtle calculations of Kohlenbach and Leuştean [18], which, in turn, draw on those in Birkhoff’s original proof. Thus the main contribution of this note is the observation that these calculations yield the stronger result stated above. The proof is fully explicit, providing a uniform, quantitative, and constructive formulation of the mean ergodic theorem for uniformly convex spaces.

In Section 2, we consider various quantitative data that can be associated to a convergence theorem, and clarify the relationship between the results here and metastability results of [1, 18, 22, 16, 17, 19]. In Section 3 we prove Theorem 1.1. Finally, in Section 4, we indicate a number of questions that remain.

## 2. QUANTITATIVE CONVERGENCE THEOREMS

Let  $(a_n)$  be a sequence of elements of a complete metric space. The next three statements all express the fact that  $(a_n)$  is convergent:

- (1) For every  $\varepsilon$ , there is an  $n$  such that for every  $i, j \geq n$ ,  $d(a_i, a_j) < \varepsilon$ .
- (2) For every  $\varepsilon$ , there is a  $k$  such that  $(a_n)$  admits at most  $k$   $\varepsilon$ -fluctuations.
- (3) For every  $\varepsilon$  and function  $g(n)$ , there is an  $n$  such that for every  $i, j \in [n, g(n)]$ ,  $d(a_i, a_j) < \varepsilon$ .

Even though the statements are equivalent, the existence assertions are quite different. A bound  $r(\varepsilon)$  on the value of  $n$  as in (1) is called a *bound on the rate of convergence* of  $(a_n)$ . A bound  $s(\varepsilon)$  on  $k$  as in (2) is called a *bound on the number of fluctuations*. A bound  $t(\varepsilon, g)$  on  $n$  as in (3) is called a *bound on the rate of metastability*.

Notice that any bound on the rate of convergence of  $(a_n)$  provides, *a fortiori*, a bound on the number of fluctuations, for every  $\varepsilon$ . Moreover, if, for every  $\varepsilon > 0$ ,  $s(\varepsilon)$  is a bound on the number of  $\varepsilon$ -fluctuations, then for any monotone function  $g$ , one of the intervals

$$[1, g(1)], [g(1), g^2(1)], \dots, [g^{s(\varepsilon)}(1), g^{s(\varepsilon)+1}(1)]$$

must fail to include a pair  $i, j$  with  $d(a_i, a_j) > \varepsilon$ . Hence  $t(\varepsilon, g) = g^{s(\varepsilon)}(1)$  is a bound on the rate of metastability.

On the other hand, it is well known in the general study of computability that there are computable, bounded, increasing sequences of rationals  $(a_n)$  that fail to have a computable rate of convergence. (Such a sequence is called a Specker sequence; see, for example, [21] or the discussion in [1, Section 5].) Clearly for such

a sequence there is a computable bound on the number of fluctuations. Similarly, it is not hard to construct a computable, bounded sequence of rationals for which there is no computable bound on the number of fluctuations. (Roughly speaking, have the sequence oscillate  $n$  times by some  $\varepsilon_n$  whenever the  $n$ th Turing machine is seen to halt on empty input.) But as long as  $g$  is computable, one can always compute a bound on  $t(\varepsilon, g)$  by searching for a suitable interval.

In a similar way, it is not hard to construct classes of sequences with a uniform bound on the number of fluctuations, but no uniform bound on the rate of convergence; and classes of sequences with a uniform bound on the rate of metastability, but no uniform bound on the number of fluctuations.<sup>1</sup>

Now consider the mean ergodic theorem, which asserts that given any nonexpansive linear operator on a Hilbert space, the averages  $(A_n x)$  converge for any elements  $x$ . It has long been known [20] that there is no uniform bound on the rate of convergence, and it is not hard to show [2, 1] that one cannot generally compute a bound on the rate of convergence from the given data. Avigad, Gerhardy, and Towsner [1] showed that there are uniform and computable bounds on the rate of metastability; see also Tao [22] for a special case, and a generalization in a different direction. That result, however, follows straightforwardly from the result of Jones, Ostrovskii, and Rosenblatt [10], stated as Theorem 1.2 above. Kohlenbach and Leuştean [18] extended the metastability result to uniformly convex Banach spaces, and Kohlenbach, Leuştean and Schade have since obtained uniform metastability bounds in much more general settings [18, 16, 17, 19]. In the next section, we show that, at least as far as the mean ergodic theorem for uniformly convex Banach spaces is considered, the stronger conclusion holds: there is a uniform and computable bound on the number of fluctuations.

(Avigad, Gerhardy, and Towsner [1] also showed that in the case of a Hilbert space, one *can* compute a bound on the rate of convergence of the ergodic averages, *given* the norm of the limit. The considerations here show that this result extends to uniformly convex Banach spaces more generally. Specifically, for every  $n, k \geq 1$  we have  $\|A_{kn}f\| = \|\frac{1}{k} \sum_{i < k} T^{in} A_n\| \leq \|A_n f\|$ , and hence the norm of the ergodic limit is the infimum of the norms  $\|A_n f\|$ . Lemma 3.1 below and the comment after the proof then shows that one can compute a rate of convergence by waiting until the norm of one of the averages is sufficiently close to this infimum.)

### 3. PROOF OF THE MAIN THEOREM

Fix a Banach space  $\mathcal{B}$  and a modulus of uniform convexity  $\eta$ . Let  $T$  be any nonexpansive linear operator on  $\mathcal{B}$ . For any element  $x$  of  $\mathcal{B}$ , we follow the notation of Kohlenbach and Leuştean [18] closely by writing  $x_n$  for the average  $A_n x$ . The following lemma is implicit in that paper:

**Lemma 3.1.** *For any  $x \in \mathcal{B}$ , let*

$$M = \left\lceil \frac{16\|x\|}{\varepsilon} \right\rceil, \quad \gamma = \frac{\varepsilon}{16} \eta \left( \frac{\varepsilon}{8\|x\|} \right).$$

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<sup>1</sup>As an example of the latter, consider the countable collection of sequences where the  $j$ th sequence starts out with 0's and then oscillates  $j$ -times from 0 to 1 or back at the  $j$ th element. For any function  $g$ , if  $g(1) < j$ , then  $[1, g(1)]$  has no oscillations; otherwise,  $g(1) \geq j$  and one of the intervals  $[g(1), g^2(1)], \dots, [g^{g(1)+1}(1), g^{g(1)+2}(1)]$  has no oscillations. Thus  $t(g, \varepsilon) = g^{g(1)+1}(1)$  is a uniform bound on the rate of metastability.

Suppose that  $N$  and  $u$  are such that for every  $m \leq u$ ,  $\|x_m\| \geq \|x_N\| - \gamma$ . Then for every pair  $i, j$  in the interval  $[MN, \lfloor u/2 \rfloor]$ , we have  $\|x_i - x_j\| < \varepsilon$ . If the modulus of convexity  $\eta$  can be written in the form  $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$  where  $\tilde{\eta}$  is nondecreasing in  $\varepsilon$ , the conclusion holds with

$$\gamma = \frac{\varepsilon}{8} \tilde{\eta} \left( \frac{\varepsilon}{8\|x\|} \right)$$

in place of the  $\gamma$  defined above.

*Proof.* This is exactly the calculation in Section 4 of Kohlenbach and Leuştean [18], with  $h(N)$  replaced by  $u$ ,  $g(MN)$  replaced by  $u/2 - MN$ , and  $b$  replaced by  $\|x\|$ .  $\square$

Notice in particular that if  $\|x_N\|$  is a “global  $\gamma$ -minimum,” which is to say,  $\|x_m\| \geq \|x_N\| - \gamma$  for every  $m$ , then Lemma 3.1 implies  $\|x_i - x_j\| < \varepsilon$  for all  $i, j > MN$ .

The next lemma shows that, in a certain sense, a small interval cannot contain many  $\varepsilon$ -fluctuations.

**Lemma 3.2.** *Fix  $N \geq 1$  and real numbers  $\alpha \geq 1$  and  $\varepsilon > 0$ . Let  $x$  be any element of  $\mathcal{B}$  such that  $\varepsilon < 2\|x\|$ . Then the number of  $\varepsilon$ -fluctuations between  $N$  and  $\alpha N$  is at most  $\lfloor 4 \log \alpha \cdot \|x\|/\varepsilon \rfloor$ .*

*Proof.* Suppose

$$N \leq i_1 \leq j_1 \leq i_2 \leq j_2 \leq \dots \leq i_s \leq j_s \leq \alpha N$$

satisfy  $\|x_{j_u} - x_{i_u}\| \geq \varepsilon$  for every  $u$ . We need to show  $s \leq 4 \log \alpha \cdot \|x\|/\varepsilon$ .

A straightforward calculation (see equation (6) of [3] or equation (11) of [18]) shows that for every  $n, k \geq 1$ ,  $\|x_{n+k} - x_k\| \leq 2k\|x\|/(n+k)$ . In particular, for any  $j = j_u$  and  $i = i_u$ , we have  $2(j-i)\|x\|/j \geq \varepsilon$ , and so

$$j \geq \left( \frac{2\|x\|}{2\|x\| - \varepsilon} \right) \cdot i = \left( 1 + \frac{\varepsilon}{2\|x\| - \varepsilon} \right) \cdot i > \left( 1 + \frac{\varepsilon}{2\|x\|} \right) \cdot i.$$

Since  $i_1 \geq N$ , we have  $j_1 \geq (1 + \varepsilon/(2\|x\|)) \cdot N$ ; since  $i_2 \geq j_1$ , we have  $j_2 \geq (1 + \varepsilon/(2\|x\|))^2 \cdot N$ , and so on. Thus  $j_s \geq (1 + \varepsilon/(2\|x\|))^s \cdot N$ . Since  $j_s \leq \alpha N$ , we have

$$(1 + (\varepsilon/(2\|x\|)))^s \leq \alpha$$

and hence

$$s \leq \frac{\log \alpha}{\log(1 + \varepsilon/(2\|x\|))} < 4 \log \alpha \cdot \|x\|/\varepsilon,$$

since  $\log(1 + \varepsilon/(2\|x\|)) > \varepsilon/(4\|x\|)$  when  $\varepsilon < 2\|x\|$ .  $\square$

Suppose we are given  $x$  in  $\mathcal{B}$  such that  $\varepsilon < 2\|x\|$ . Consider the sequence  $x_1, x_2, x_3, \dots$  of averages. Let  $M$  and  $\gamma$  be as in the statement of Lemma 3.1. Define a finite sequence  $N_0, N_1, \dots, N_s$  of integers by setting  $N_0 = 1$  and setting  $N_{i+1}$  equal to the least  $m$  such that  $\|x_m\| < \|x_{N_i}\| - \gamma$ , if such an  $m$  exists. Notice that  $N_{i+1} \geq N_i$  for every  $i < s$ . Notice also that  $s \leq \lfloor \|x\|/\gamma \rfloor$ , as the norm of the averages cannot drop by  $\gamma$  more than  $\lfloor \|x\|/\gamma \rfloor$ -many times.

The idea is this: Lemma 3.1 tells us that there are no  $\varepsilon$ -fluctuations in the intervals  $[MN_0, N_1/2)$ ,  $[MN_1, N_2/2)$ ,  $\dots$ ,  $[MN_{s-1}, N_s/2)$ , or beyond  $MN_s$ . This leaves the  $\varepsilon$ -fluctuations in the intervals  $[1, MN_0)$  and  $[N_u/2, MN_u)$  for  $u = 1, \dots, s$ ,

whose number we can bound using Lemma 3.2; as well as at most  $s$ -many  $\varepsilon$ -fluctuations that span more than one interval. (The fact that the intervals in the first sentence may overlap or that some of the  $N_u$ 's may be odd does not hurt the argument below.)

More precisely, suppose  $i_1 \leq j_1 \leq \dots \leq i_k \leq j_k$  are such that for each  $u = 1, \dots, k$ ,  $\|a_{j_u} - a_{i_u}\| \geq \varepsilon$ . Lemma 3.2 tells us that at most  $\lfloor 4 \log M \cdot \|x\|/\varepsilon \rfloor$  of the pairs  $(i_u, j_u)$  lie in the first interval,  $[1, MN_0) = [1, M)$ , and at most  $\lfloor 4 \log(2M) \cdot \|x\|/\varepsilon \rfloor$  of them lie in the remaining ones. Each of the remaining pairs has to straddle at least one of the  $N_u$ 's for  $u = 1, \dots, s$ . Thus  $k$  is at most

$$\left\lfloor 4 \log M \cdot \frac{\|x\|}{\varepsilon} \right\rfloor + \left\lfloor \frac{\|x\|}{\gamma} \right\rfloor \cdot \left\lfloor 4 \log(2M) \cdot \frac{\|x\|}{\varepsilon} \right\rfloor + \left\lfloor \frac{\|x\|}{\gamma} \right\rfloor.$$

This provides a precise bound on the number of fluctuations, but some simplification will improve readability.

If  $\|x\|/\varepsilon$  is sufficiently large, we can expand the definitions of  $M$  and  $\gamma$  and absorb the first and third terms and various constants into a constant multiple of the second term. More precisely, setting  $\rho = \|x\|/\varepsilon$ , the sequence of ergodic averages admits at most  $O(\rho^2 \log \rho \cdot \eta(1/(8\rho))^{-1})$ -many  $\varepsilon$ -fluctuations. When the additional criterion of Lemma 3.1 is met, we can replace  $\eta$  by  $\tilde{\eta}$  in this statement.

This concludes the proof of Theorem 1.1. As noted by Kohlenbach and Leuştean [18, Corollary 2.3], in the case of a Hilbert space we can take  $\tilde{\eta}(\varepsilon) = \varepsilon/8$ . In that case, Theorem 1.1 ensures that the sequence of ergodic averages admits at most  $O(\rho^3 \log \rho)$ -many  $\varepsilon$ -fluctuations, but this is weaker than the bound of  $O(\rho^2)$  that can be obtained from Theorem 1.2.

As noted in Section 2, Theorems 1.1 and Theorems 1.2 yield uniform and computable bounds on the rate of metastability as well. In the case of a uniformly convex Banach space, the more direct argument by Kohlenbach and Leuştean yields a bound that is quantitatively better, requiring only  $O(\rho \log \rho \cdot \eta(1/(8\rho))^{-1})$ -many iterations of a function that grows slightly faster than the argument  $g$  described in Section 2.<sup>2</sup>

#### 4. COMMENTS AND QUESTIONS

We have observed that saying that a sequence of elements of a complete metric space converges is equivalent to saying that, for every  $\varepsilon > 0$ , the sequence admits only finitely many  $\varepsilon$ -fluctuations. Similarly, if  $(f_n)$  is a sequence of measurable functions from a measure space  $\mathcal{X} = (X, \mathcal{B}, \mu)$  to some metric space, saying that  $(f_n)$  converges pointwise a.e. is equivalent to saying that for every  $\varepsilon > 0$ , the measure of the set

$$\{x \in X \mid (f_n(x))_{n \in \mathbb{N}} \text{ admits } k \text{ } \varepsilon\text{-fluctuations}\}$$

approaches 0 as  $k$  approaches infinity. Such results can often be obtained from upcrossing inequalities in the style of Doob's upcrossing inequality for the martingale convergence theorem [7] and Bishop's upcrossing inequalities for the pointwise ergodic theorem and Lebesgue's theorem [4, 5, 6]. Other upcrossing inequalities and oscillation inequalities have been obtained in the measure-theoretic setting [9, 13, 11, 12, 14, 8].

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<sup>2</sup>Note that due to an error in typesetting there is an extra "h" in the statement of the main theorem in [18].

Theorems 1.1 and 1.2 show that the uniform bound on the oscillations of a sequence of ergodic averages is a geometric rather than metric phenomenon. To summarize the state of affairs:

- For Hilbert spaces, Jones, Ostrovskii, and Rosenblatt [10] provide a square-function inequality (Theorem 1.2 above), which immediately yields a uniform and explicit bound on the number of  $\varepsilon$ -fluctuations in a sequence of ergodic averages.
- In the setting of uniformly convex Banach spaces, our Theorem 1.1 provides a uniform and explicit bound on the number of  $\varepsilon$ -fluctuations.
- Kohlenbach, Leuştean, and Schade [16, 17, 19] provide uniform bounds on the rate of metastability for more generalized averaging schemes, in more general classes of Banach spaces.

The following questions remain:

- Can one obtain something like the square-function inequality of [10] for uniformly convex spaces?
- Can one obtain a sharper bound on the number of  $\varepsilon$ -fluctuations in the uniformly convex setting, in particular one which is  $O(\rho^2)$  when specialized to Hilbert spaces?
- To what extent can uniform bounds on the number of  $\varepsilon$ -fluctuations be obtained in more general ergodic theorems?
- In particular, are there uniform bounds on the number of  $\varepsilon$ -fluctuations for sequences of multiple ergodic averages, as in Tao’s convergence theorem [22]?

Finally, it is worth noting that Kohlenbach’s “proof mining” program [15] provides general logical methods for extracting bounds on the rate of metastability, as well as general conditions that guarantee the relevant uniformities (see also the discussions in [1, 16, 17]). Along these lines, it would be nice to have a better general understanding as to when (and how) uniform and computable bounds on the number of fluctuations can be obtained from a nonconstructive convergence theorem.

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